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Generalized Dirichlet to Neumann operator on invariant differential forms and equivariant cohomology

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ABSTRACT

In recent work, Belishev and Sharafutdinov show that the generalized Dirichlet to Neumann (DN) operator Λ on a compact Riemannian manifold M with boundary ∂M determines de Rham cohomology groups of M. In this paper, we suppose G is a torus acting by isometries on M. Given X in the Lie algebra of G and the corresponding vector field X_M on M, Witten defines an inhomogeneous coboundary operator $\mathrm{d}_{X_M} = \mathrm{d} + \iota_{X_M}$ on invariant forms on M. The main purpose is to adapt Belishev–Sharafutdinov's boundary data to invariant forms in terms of the operator d_{X_M} in order to investigate to what extent the equivariant topology of a manifold is determined by the corresponding variant of the DN map. We define an operator Λ_{X_M} on invariant forms on the boundary which we call the X_M-DN map and using this we recover the X_M-cohomology groups from the generalized boundary data $(\partial M, \Lambda_{X_M})$. This shows that for a Zariski-open subset of the Lie algebra, Λ_{X_M} determines the free part of the relative and absolute equivariant cohomology groups of M. In addition, we partially determine the ring structure of X_M-cohomology groups from Λ_{X_M}. These results explain to what extent the equivariant topology of the manifold in question is determined by Λ_{X_M}.

1. Introduction

The classical Dirichlet to Neumann (DN) operator $\Lambda_{cl} : C^\infty(\partial M) \longrightarrow C^\infty(\partial M)$ is defined by $\Lambda_{cl}\theta = \partial\omega/\partial\nu$, where ω is the solution to the Dirichlet problem

$$\Delta\omega = 0, \qquad \omega|_{\partial M} = \theta$$

and ν is the unit outer normal to the boundary. In the scope of inverse problems of reconstructing a manifold from the boundary measurements, the following question is of great theoretical and applied interest [7]: *to what extent are the topology and geometry of M determined by the DN operator*?

In this paper we are interested in the equivariant topology analogue of this question.

Much effort has been made to address this (non-equivariant) question. For instance, in the case of a two-dimensional manifold M with a connected boundary, an explicit formula is obtained which expresses the Euler characteristic of M in terms of Λ_{cl} and the Euler characteristic completely determines the topology of M in this case [6]. In the three-dimensional case [5], some formulas are obtained which express the Betti numbers $\beta_1(M)$ and $\beta_2(M)$ in terms of Λ_{cl} and their operator on vector fields, $\vec{\Lambda} : C^\infty(T(\partial M)) \longrightarrow C^\infty(T(\partial M))$. This culminates in recent work of Belishev and Sharafutdinov [7] who prove that the real additive de Rham cohomology of a compact, connected, oriented smooth Riemannian manifold M of

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dimension n with boundary is completely determined by its boundary data $(\partial M, \Lambda)$ where $\Lambda : \Omega^k(\partial M) \longrightarrow \Omega^{n-k-1}(\partial M)$ is a generalization of the classical Dirichlet to Neumann operator Λ_{cl} to the space of differential forms. More precisely, they define the DN operator Λ as follows [7]: given $\theta \in \Omega^k(\partial M)$, the boundary value problem

$$\Delta\omega = 0, \qquad i^*\omega = \theta, \qquad i^*(\delta\omega) = 0 \tag{1.1}$$

is solvable and the operator Λ is given by the formula $\Lambda\theta = i^*(\star \mathsf{d}\omega)$, where i^* is the pullback by the inclusion map $i : \partial M \hookrightarrow M$. Here δ is the formal adjoint of d relative to the L^2-inner product

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge (\star\beta)$$

which is defined on $\Omega^k(M)$, and $\star : \Omega^k \longrightarrow \Omega^{n-k}$ is the Hodge star operator.

More concretely, there are two distinguished finite dimensional subspaces of $\mathcal{H}^k(M) = \ker \mathsf{d} \cap \ker \delta \subset \Omega^k(M)$, whose elements are called Dirichlet and Neumann harmonic fields respectively, namely

$$\mathcal{H}^k_D(M) = \{\lambda \in \mathcal{H}^k(M) \mid i^*\lambda = 0\}, \qquad \mathcal{H}^k_N(M) = \{\lambda \in \mathcal{H}^k(M) \mid i^*\star\lambda = 0\}.$$

The dimensions of these spaces are given by $\dim \mathcal{H}^k_D(M) = \dim \mathcal{H}^{n-k}_N(M) = \beta_k(M)$, where $\beta_k(M)$ is the kth Betti number [11]. They prove the following theorem:

Theorem 1.1. *(Belishev and Sharafutdinov [7]) For any $0 \leqslant k \leqslant n-1$, the range of the operator*

$$\Lambda + (-1)^{nk+k+n} \mathsf{d}\Lambda^{-1}\mathsf{d} : \Omega^k(\partial M) \longrightarrow \Omega^{n-k-1}(\partial M)$$

is $i^*\mathcal{H}^{n-k-1}_N(M)$.

Since $i^*\mathcal{H}^k_N(M) \cong \mathcal{H}^k_N(M) \cong H^k(M)$, it follows that $(\Lambda + (-1)^{nk+k+1}\mathsf{d}\Lambda^{-1}\mathsf{d})\Omega^{n-k}(\partial M) \cong H^k(M)$. Using, Poincaré–Lefschetz duality, $H^k(M) \cong H^{n-k}(M, \partial M)$, the theorem immediately implies that the data $(\partial M, \Lambda)$ determines both the absolute and relative de Rham cohomology groups.

In addition, they present the following theorem which gives the lower bound for the Betti numbers of the manifold M and its boundary through the DN operator Λ.

Theorem 1.2. *(Belishev and Sharafutdinov [7]) The kernel of Λ contains the space $\mathcal{E}(\partial M)$ of exact forms and for each k,*

$$\dim[\ker \Lambda^k / \mathcal{E}^k(\partial M)] \leqslant \min\{\beta_k(\partial M), \beta_k(M)\}$$

where $\beta_k(\partial M)$ and $\beta_k(M)$ are the Betti numbers, and Λ^k is the restriction of Λ to $\Omega^k(\partial M)$.

At the end of their paper, they posed the following problem: *can the multiplicative structure of the cohomologies be recovered from the data $(\partial M, \Lambda)$?*

To give a partial answer to this question, Shonkwiler [12, Sect. 5.3] defines the map

$$(\phi, \psi) \longmapsto (-1)^k \Lambda(\phi \wedge \Lambda^{-1}\psi), \quad \forall (\phi, \psi) \in i^*\mathcal{H}^k_N(M) \times i^*\star\mathcal{H}^l_D(M). \tag{1.2}$$

More precisely, by using the classical wedge product between the differential forms, he considers the mixed cup product between the absolute cohomology $H^k(M, \mathbb{R})$ and the relative cohomology $H^l(M, \partial M, \mathbb{R})$, i.e.

$$\cup : H^k(M, \mathbb{R}) \times H^l(M, \partial M, \mathbb{R}) \longrightarrow H^{k+l}(M, \partial M, \mathbb{R})$$

and then he restricts the second argument to come from the *boundary subspace*. This subspace is defined by DeTurck and Gluck [9] as the subspace of $H^k(M, \partial M)$ consisting of exact forms which satisfy the Dirichlet boundary condition (i.e. i^* of these exact forms are zero). Shonkwiler then presents the following partial answer to Belishev and Sharafutdinov's question:

Theorem 1.3. *(Shonkwiler [12]) The boundary data $(\partial M, \Lambda)$ completely determines the mixed cup product in terms of the map* (1.2) *when the relative cohomology class is restricted to belong to the boundary subspace.*

Equivariant setting We briefly review some notation and results from [2]. Let M be a compact, oriented, smooth Riemannian manifold with boundary and suppose G is a torus acting by isometries on M. Denote by Ω^k_G the k-forms invariant under the G-action. Given X in the Lie algebra $\mathfrak{g}$ of G and corresponding vector field X_M on M, consider Witten's coboundary operator $\mathsf{d}_{X_M} = \mathsf{d} + \iota_{X_M}$. This operator is no longer homogeneous in the degree of the smooth invariant form on M: if $\omega \in \Omega^k_G$ then $\mathsf{d}_{X_M}\omega \in \Omega^{k+1}_G \oplus \Omega^{k-1}_G$. Note then that $\mathsf{d}_{X_M} : \Omega^\pm_G \longrightarrow \Omega^\mp_G$, where $\Omega^\pm_G$ is the space of invariant forms of even $(+)$ or odd $(-)$ degree. Let δ_{X_M} be the adjoint of d_{X_M} and define the resulting *Witten–Hodge–Laplacian* to be $\Delta_{X_M} = (\mathsf{d}_{X_M} + \delta_{X_M})^2 = \mathsf{d}_{X_M}\delta_{X_M} + \delta_{X_M}\mathsf{d}_{X_M}$.

Because the forms are invariant, it is easy to see that $d_{X_M}^2 = 0$ (see [2] for details). In this setting, we define two types of X_M-cohomology, the absolute X_M-cohomology $H^{\pm}_{X_M}(M)$ and the relative X_M-cohomology $H^{\pm}_{X_M}(M, \partial M)$. The first is the cohomology of the complex (Ω_G, d_{X_M}), while the second is the cohomology of the subcomplex $(\Omega_{G,D}, d_{X_M})$, where $\omega \in \Omega^{\pm}_{G,D}$ if it satisfies $i^*\omega = 0$ (the D is for Dirichlet boundary condition). One also defines $\Omega^{\pm}_{G,N}(M) = \{\alpha \in \Omega^{\pm}_G(M) \mid i^*(\star\alpha) = 0\}$ (Neumann boundary condition). Clearly, the Hodge star $\star$ provides an isomorphism $\Omega^{\pm}_{G,D} \cong \Omega^{n-\pm}_{G,N}$, where we write $n - \pm$ for the parity (modulo 2) resulting from subtracting an even/odd number from n. Furthermore, because d_{X_M} and i^* commute, it follows that d_{X_M} preserves the Dirichlet boundary conditions while δ_{X_M} preserves Neumann boundary conditions. Because of boundary terms, the null space of Δ_{X_M} does not coincide with the closed and co-closed forms in Witten's sense. Elements of $\ker \Delta_{X_M}$ are called *X_M-harmonic forms* while the ω which satisfy $d_{X_M}\omega = \delta_{X_M}\omega = 0$ are *X_M-harmonic fields*; it is clear that every X_M-harmonic field is an X_M-harmonic form, but the converse is false. The infinite dimensional space of X_M-harmonic *fields* is denoted $\mathcal{H}^{\pm}_{X_M}(M)$, so we have $\mathcal{H}^{*}_{X_M}(M) \subset \ker \Delta_{X_M}$. Two useful finite dimensional subspaces of $\mathcal{H}^{\pm}_{X_M}(M)$ are the Dirichlet and Neumann X_M-harmonic fields, respectively: $\mathcal{H}^{\pm}_{X_M,D}(M)$ and $\mathcal{H}^{\pm}_{X_M,N}(M)$. There are therefore two different candidates for X_M-harmonic representatives when the boundary is present. This construction firstly leads us to present the X_M-Hodge–Morrey decomposition theorem which states that

$$\Omega^{\pm}_G(M) = \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M}(M) \tag{1.3}$$

where $\mathcal{E}^{\pm}_{X_M}(M) = \{d_{X_M}\alpha \mid \alpha \in \Omega^{\mp}_{G,D}\}$ and $\mathcal{C}^{\pm}_{X_M}(M) = \{\delta_{X_M}\beta \mid \beta \in \Omega^{\mp}_{G,N}\}$. This decomposition is orthogonal with respect to the L^2-inner product given above.

In addition, in [2] we present an X_M-Friedrichs Decomposition Theorem which states that

$$\begin{aligned} \mathcal{H}^{\pm}_{X_M}(M) &= \mathcal{H}^{\pm}_{X_M,D}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M), \\ \mathcal{H}^{\pm}_{X_M}(M) &= \mathcal{H}^{\pm}_{X_M,N}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) \end{aligned} \tag{1.4}$$

where $\mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) = \{\xi \in \mathcal{H}^{\pm}_{X_M}(M) \mid \xi = d_{X_M}\sigma\}$ and $\mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) = \{\eta \in \mathcal{H}^{\pm}_{X_M}(M) \mid \eta = \delta_{X_M}\alpha\}$. Together these give the orthogonal *X_M-Hodge–Morrey–Friedrichs* decompositions [2],

$$\begin{aligned} \Omega^{\pm}_G(M) &= \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,D}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) \\ &= \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,N}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M). \end{aligned} \tag{1.5}$$

The two decompositions are related by the Hodge star operator. The orthogonality of (1.3)–(1.5) follows from Green's formula for d_{X_M} and δ_{X_M} which states

$$\langle d_{X_M}\alpha, \beta\rangle = \langle \alpha, \delta_{X_M}\beta\rangle + \int_{\partial M} i^*(\alpha \wedge \star\beta) \tag{1.6}$$

for all $\alpha, \beta \in \Omega_G$.

The consequence for X_M-cohomology is that each class in $H^{\pm}_{X_M}(M)$ is represented by a unique X_M-harmonic field in $\mathcal{H}^{\pm}_{X_M,N}(M)$, and each relative class in $H^{\pm}_{X_M}(M, \partial M)$ is represented by a unique X_M-harmonic field in $\mathcal{H}^{\pm}_{X_M,D}(M)$. We also elucidate in [2] the connection between the X_M-cohomology groups and the free part of the relative and absolute equivariant cohomology groups.

The *X_M-Hodge–Morrey–Friedrichs* decompositions (1.5) of smooth invariant differential forms give us insight to create boundary data which is a generalization of Belishev and Sharafutdinov's boundary data on $\Omega^{\pm}_G(\partial M)$.

In this paper, we take a topological approach, looking to determine the X_M-cohomology groups and the free part of the equivariant cohomology groups from the generalized boundary data. To this end, in Section 2 we prove that the concrete realizations $\mathcal{H}^{\pm}_{X_M,N}(M)$ and $\mathcal{H}^{\pm}_{X_M,D}(M)$ of the absolute and relative X_M-cohomology groups respectively meet only at the origin and in Section 3 we define the X_M-DN operator Λ_{X_M} on $\Omega^{\pm}_G(\partial M)$, the definition involves showing that certain boundary value problems are solvable. The definition of Λ_{X_M} represents a generalization of Belishev and Sharafutdinov's DN operator Λ on $\Omega^{\pm}_G(\partial M)$ in the sense that when $X = 0$, we have $\Lambda_0 = \Lambda$. Finally, in the remaining sections, we explain how the boundary data $(\partial M, \Lambda_{X_M})$ encodes more information about the equivariant algebraic topology of M than does the boundary data $(\partial M, \Lambda)$ on ∂M. Hence, these results contribute to explain to what extent the equivariant topology of the manifold in question is determined by the X_M-DN map Λ_{X_M}.

Throughout this paper, when arguments follow closely the corresponding arguments in the non-equivariant setting we refer to the original argument and omit the details. These details can be found in the first author's thesis [1].

2. Main results

Throughout we let M be a compact, connected, oriented, smooth Riemannian manifold with boundary and we suppose G is a torus acting by isometries on M. Given X in the Lie algebra $\mathfrak{g}$ and corresponding vector field X_M on M, one defines

Witten's inhomogeneous coboundary operator $\mathrm{d}_{X_M} = \mathrm{d} + \iota_{X_M} : \Omega_G^{\pm} \longrightarrow \Omega_G^{\mp}$ and the resulting X_M-harmonic fields and forms as described in the introduction.

An important classical result is that any harmonic field satisfying both Neumann and Dirichlet boundary conditions (so one vanishing on the boundary) is necessarily zero: see Theorem 3.4.4 in [11] or Lemma 2 in [8].

Theorem 2.1. *If an X_M-harmonic field $\lambda \in \mathcal{H}^{\pm}_{X_M}(M)$ vanishes on the boundary ∂M, then $\lambda \equiv 0$, i.e.*

$$\mathcal{H}^{\pm}_{X_M,N}(M) \cap \mathcal{H}^{\pm}_{X_M,D}(M) = \{0\}. \tag{2.1}$$

The proof consists in showing that a harmonic field which is both Neumann and Dirichlet has a zero of infinite order at every boundary point and then applying the Strong Unique Continuation Theorem below. However, the proof that there are zeros of infinite order in [11,8] does not appear to extend to our present setting, so we give a different argument, based on Hadamard's lemma, and which is also valid in the classical case.

First, we state the Strong Unique Continuation Theorem, due to Aronszajn [3], Aronszajn, Krzywicki and Szarski [4]. In [10], Kazdan writes this theorem in terms of Laplacian operator Δ but he mentions that it is still valid for any operator having the diagonal form $P = \Delta I +$ lower-order terms, where I is the identity matrix. Hence, one can state this theorem in terms of diagonal form operator by the following form:

Theorem 2.2 *(Strong Unique Continuation Theorem). ([10]) Let $\overline{M}$ be a Riemannian manifold with Lipschitz continuous metric, and let ω be a differential form having first derivatives in L^2 that satisfies $P(\omega) = 0$ where P is a diagonal form operator. If ω has a zero of infinite order at some point in $\overline{M}$, then ω is identically zero on $\overline{M}$.*

Proof of Theorem 2.1. Suppose $\lambda \in \mathcal{H}^{\pm}_{X_M,N}(M) \cap \mathcal{H}^{\pm}_{X_M,D}(M)$, then λ is smooth by using the results of [2]. Since $i^*\lambda = i^* \star \lambda = 0$ then $\lambda|_{\partial M} \equiv 0$ and we have that $(\iota_{X_M}\lambda)|_{\partial M} = 0$ as well.

The proof is local so we can consider M to be the upper half space in $\mathbb{R}^n$ with $\partial M = \mathbb{R}^{n-1}$. Since the metric, the differential form λ and the vector field X_M are given in the upper half space, we can extend them from there to all of $\mathbb{R}^n$ by reflection in $\partial M = \mathbb{R}^{n-1}$. The resulting objects are: the extended metric, which will be Lipschitz continuous [8]; we extend λ to all of $\mathbb{R}^n$ by making it odd with respect to reflection in $\mathbb{R}^{n-1}$ and extend X_M to all of $\mathbb{R}^n$ by making it even with respect to reflection in $\mathbb{R}^{n-1}$ and the extended X_M will be a Lipschitz continuous vector field. But the original λ satisfies $\lambda|_{\partial M} \equiv 0$ and $\mathrm{d}_{X_M}\lambda = \delta_{X_M}\lambda = 0$ on the upper half space, hence the extended one will be of class C^1 and satisfy $\mathrm{d}_{X_M}\lambda = \delta_{X_M}\lambda = 0$ on $\mathbb{R}^n$, i.e. the extended λ satisfies $P(\lambda) = \Delta_{X_M}\lambda = 0$ on all of $\mathbb{R}^n$ where the operator Δ_{X_M} has diagonal form, i.e. $P = \Delta_{X_M} = \Delta I +$ lower-order terms. So far, we have satisfied the first condition of Theorem 2.2.

Now, we need to satisfy the remaining hypotheses of Theorem 2.2. Let $x = (x', x_n)$ be a coordinate chart where $x' = (x_1, x_2, \ldots, x_{n-1})$ is a chart on the boundary ∂M and x_n is the distance to the boundary. In these coordinates $x_n > 0$ in M and ∂M is locally characterized by $x_n = 0$. These coordinates are called boundary normal coordinates and the Riemannian metric in these coordinates has the form $\sum_{m,r=1}^{n-1} h_{m,r}(x)\mathrm{d}x^m \otimes \mathrm{d}x^r + \mathrm{d}x^n \otimes \mathrm{d}x^n$.

Now consider a neighborhood of $p \in \partial M$ where the boundary normal coordinates are well defined. We can write $\lambda = \alpha + \beta \wedge \mathrm{d}x_n$ where $\alpha = \sum f_{I_1}(x)\mathrm{d}x^{I_1}$, $\beta = \sum g_{I_2}(x)\mathrm{d}x^{I_2}$ and $I_1, I_2 \subset \{1, 2, \ldots, n-1\}$. Our goal is to prove that all the partial derivatives of the coefficients of λ (i.e. of $f_{I_1}(x)$ and $g_{I_2}(x)$) vanish at $p \in \partial M$. Now, $\lambda|_{\partial M} \equiv 0$ which implies that $f_{I_1}(x', 0) = g_{I_2}(x', 0) = 0$. Hence, we can apply Hadamard's lemma to $f_{I_1}(x)$ and $g_{I_2}(x)$ and write $f_{I_1}(x) = x_n \overline{f}_{I_1}(x)$ and $g_{I_2}(x) = x_n \overline{g}_{I_2}(x)$ for some smooth functions $\overline{f}_{I_1}(x)$ and $\overline{g}_{I_2}(x)$. Moreover, these representations for $f_{I_1}(x)$ and $g_{I_2}(x)$ imply that all the higher partial derivatives of $f_{I_1}(x)$ and $g_{I_2}(x)$ with respect to each of the x'-coordinates (i.e. except the normal direction coordinate x_n) at the point p are zero.

Therefore, we only need to prove that all the higher partial derivatives of $f_{I_1}(x)$ and $g_{I_2}(x)$ in the normal direction are zero to deduce that the Taylor series of $f_{I_1}(x)$ and $g_{I_2}(x)$ around $x_n = 0$ are zero. The proof of this is by contradiction.

Suppose the Taylor series of $f_{I_1}(x)$ and $g_{I_2}(x)$ around $x_n = 0$ are not zero at $p \in \partial M$ which means that there exist the largest positive integer numbers k and j such that $f_{I_1}(x) = x_n^k \widehat{f}_{J_1}(x)$ and $g_{I_2}(x) = x_n^j \widehat{g}_{J_2}(x)$ where $\widehat{f}_{J_1}(x', 0) \neq 0$ and $\widehat{g}_{J_2}(x', 0) \neq 0$ for some J_1, J_2. Thus, we can always write λ in the following form $\lambda = x_n^k \tau + x_n^j \rho \wedge \mathrm{d}x_n$ where the differential forms τ and ρ do not contain $\mathrm{d}x_n$. Applying $\mathrm{d}_{X_M}\lambda = 0$, we get

$$0 = \mathrm{d}_{X_M}\lambda = k x_n^{k-1}\mathrm{d}x_n \wedge \tau + x_n^k \mathrm{d}\tau + x_n^j \mathrm{d}\rho \wedge \mathrm{d}x_n + x_n^k \iota_{X_M}\tau + x_n^j \iota_{X_M}(\rho \wedge \mathrm{d}x_n).$$

Now, reducing this equation modulo x_n^k we conclude that the term $x_n^j(\mathrm{d}\rho \wedge \mathrm{d}x_n + \iota_{X_M}(\rho \wedge \mathrm{d}x_n)) \not\equiv 0$ modulo x_n^k because the term $k x_n^{k-1}\mathrm{d}x_n \wedge \tau \not\equiv 0$ modulo x_n^k and as a consequence, we infer that $k > j$.

Similarly, we can calculate $\delta_{X_M}\lambda = -(\mp)^n(\star \mathrm{d} \star \lambda + \star \iota_{X_M} \star \lambda) = 0$ using the Riemannian metric above. It suffices to use $\mathrm{d} \star \lambda + \iota_{X_M} \star \lambda = 0$, where $\star\lambda = x_n^k \xi \wedge \mathrm{d}x_n + x_n^j \zeta$ for differential forms ξ and ζ which do not contain $\mathrm{d}x_n$ (both of them will contain many of the coefficients $h_{m,r}(x)$). Hence, we get

$$0 = \mathrm{d} \star \lambda + \iota_{X_M} \star \lambda = x_n^k \mathrm{d}\xi \wedge \mathrm{d}x_n + j x_n^{j-1}\mathrm{d}x_n \wedge \zeta + x_n^j \mathrm{d}\zeta + x_n^k \iota_{X_M}(\xi \wedge \mathrm{d}x_n) + x_n^j \iota_{X_M}\zeta.$$

Reducing this equation modulo x_n^j and for the same reason above but replacing k by j, we can infer that $k < j$. But this is a contradiction, so there are no such largest positive integers k and j. Hence, the Taylor series for the coefficients $f_{I_1}(x)$ and $g_{I_2}(x)$ around $x_n = 0$ must be zero at $p \in \partial M$. It means that all the higher partial derivatives of $f_{I_1}(x)$ and $g_{I_2}(x)$ vanish at all points of the boundary ∂M. Thus, these facts are enough to show all mixed partial derivatives including x_n also vanish at the boundary. Hence, λ has a zero of infinite order at $p \in \partial M$.

The remaining possibility of one of the coefficients $f_{I_1}(x)$ and $g_{I_2}(x)$ having finite order and the other infinite order in x_n follows from the same argument as above.

Thus, λ satisfies all the hypotheses of the Strong Unique Continuation Theorem 2.2, so must be zero on all of $\mathbb{R}^n$. Since M is assumed to be connected, λ must be identically zero on all of M. $\square$

As a consequence of Theorem 2.1, we obtain the following.

Corollary 2.3.

(1) *The space of X_M-harmonic fields can be written as a (not direct) sum*:
$$\mathcal{H}^{\pm}_{X_M}(M) = \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) + \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M). \tag{2.2}$$
(2) *The trace map $i^* : \mathcal{H}^{\pm}_{X_M,N}(M) \longrightarrow i^*\mathcal{H}^{\pm}_{X_M,N}(M)$ is an isomorphism.*
(3) *The map $f : i^*\mathcal{H}^{\pm}_{X_M,N}(M) \longrightarrow H^{\pm}_{X_M}(M)$ defined by $f(i^*\lambda_N) = [\lambda_N]$ for $\lambda_N \in \mathcal{H}^{\pm}_{X_M,N}(M)$ is an isomorphism.*
(4) *The map $h : i^*\mathcal{H}^{n-\pm}_{X_M,N}(M) \longrightarrow H^{\pm}_{X_M}(M, \partial M)$ defined by $h(i^*\lambda_N) = [\star\lambda_N]$ for $\lambda_N \in \mathcal{H}^{n-\pm}_{X_M,N}(M)$ is an isomorphism.*

Proof. (1) This follows by applying Theorem 2.1 and the X_M-Friedrichs Decomposition (1.4).

(2) It is clear that i^* is surjective and it follows from Theorem 2.1 that it is injective.

(3) f is a well-defined map because $\ker i^* = \{0\}$. Furthermore, f is a bijection because there exists a unique Neumann X_M-harmonic field in any absolute X_M-cohomology class (Corollary 3.17 of [2]) hence part (3) holds.

(4) This follows from part (3) by using X_M-Poincaré–Lefschetz duality of [2] (i.e. $H^{\pm}_{X_M}(M) \cong H^{n-\pm}_{X_M}(M, \partial M)$). $\square$

3. X_M-DN operator

Before defining this operator, we first need to prove the solvability of a certain boundary value problem (3.1). The proof depends on the main results in [2] and there is not any corresponding statement of it in [11]. When $X = 0$, this gives an independent proof of the solvability of Belishev and Sharafutdinov's BVP (1.1). Theorem 3.1 represents the keystone to defining the X_M-DN operator and then to exploiting a connection between this X_M-DN operator and X_M-cohomology via the Neumann X_M-trace space $i^*\mathcal{H}^{\pm}_{X_M,N}(M)$.

Theorem 3.1. *Given $\theta \in \Omega^{\pm}_G(\partial M)$ and $\eta \in \Omega^{\pm}_G(M)$, then the* BVP
$$\begin{cases} \Delta_{X_M}\omega = \eta & \text{on } M, \\ i^*\omega = \theta & \text{on } \partial M, \\ i^*(\delta_{X_M}\omega) = 0 & \text{on } \partial M \end{cases} \tag{3.1}$$
is solvable for $\omega \in \Omega^{\pm}_G(M)$ if and only if
$$\langle \eta, \kappa_D \rangle = 0, \quad \forall \kappa_D \in \mathcal{H}^{\pm}_{X_M,D}(M). \tag{3.2}$$
The solution of BVP (3.1) *is unique up to an arbitrary Dirichlet X_M-harmonic field $\mathcal{H}^{\pm}_{X_M,D}(M)$.*

Proof. Suppose BVP (3.1) has a solution. Then one can easily show that condition (3.2) holds by using Green's formula (1.6).

Now suppose $\eta \in \Omega^{\pm}_G(\partial M)$ satisfies $\langle \eta, \kappa_D \rangle = 0$, $\forall \kappa_D \in \mathcal{H}^{\pm}_{X_M,D}(M)$ (i.e. $\eta \in \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$). Since $\theta \in \Omega^{\pm}_G(\partial M)$, we can construct an extension $\omega_1 \in \Omega^{\pm}_G(M)$ of the differential form $\theta \in \Omega^{\pm}_G(\partial M)$ such that
$$i^*\omega_1 = \theta, \quad \omega_1 = \delta_{X_M}\beta_{\omega_1} + \lambda_{\omega_1} \in \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M}(M).$$
But $\Delta_{X_M}\omega_1 = \delta_{X_M} \mathrm{d}_{X_M} \delta_{X_M}\beta_{\omega_1}$, so (1.6) implies that $\Delta_{X_M}\omega_1 \in \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$ as well. Hence, $\eta - \Delta_{X_M}\omega_1 \in \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$. We now apply Proposition 3.8 of [2] which for smooth invariant forms states that for each $\overline{\eta} \in \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$ there is a unique smooth differential form $\overline{\omega} \in \Omega^{\pm}_{G,D} \cap \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$ satisfying the BVP (3.1) but with $\eta = \overline{\eta}$ and $\theta = 0$. Since $\eta - \Delta_{X_M}\omega_1 \in \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$ is smooth, it follows from this that there is a unique smooth differential form $\omega_2 \in \Omega^{\pm}_{G,D} \cap \mathcal{H}^{\pm}_{X_M,D}(M)^{\perp}$ which satisfies the BVP
$$\begin{cases} \Delta_{X_M}\omega_2 = \eta - \Delta_{X_M}\omega_1 & \text{on } M, \\ i^*\omega_2 = 0 & \text{on } \partial M, \\ i^*(\delta_{X_M}\omega_2) = 0 & \text{on } \partial M. \end{cases} \tag{3.3}$$

Now, let $\omega_2 = \omega - \omega_1$, then the BVP (3.3) turns into the BVP (3.1). Hence, there exists a solution to the BVP (3.1) which is $\omega = \omega_1 + \omega_2$, where the uniqueness of ω is up to an arbitrary Dirichlet X_M-harmonic field. $\square$

Definition 3.2 *(X_M-DN operator Λ_{X_M}).* We consider on M the BVP (3.1) with $\eta = 0$, i.e.

$$\begin{cases} \Delta_{X_M}\omega = 0 & \text{on } M, \\ i^*\omega = \theta & \text{on } \partial M, \\ i^*(\delta_{X_M}\omega) = 0 & \text{on } \partial M \end{cases} \tag{3.4}$$

then by Theorem 3.1 BVP (3.4) is solvable and the solution is unique up to an arbitrary Dirichlet X_M-harmonic field $\kappa_D \in \mathcal{H}^{\pm}_{X_M,D}(M)$. We can therefore define the X_M-DN operator $\Lambda_{X_M} : \Omega^{\pm}_G(\partial M) \longrightarrow \Omega^{n-\mp}_G(\partial M)$ by

$$\Lambda_{X_M}\theta = i^*(\star\, \mathrm{d}_{X_M}\omega).$$

Note that taking $\mathrm{d}_{X_M}\omega$ eliminates the ambiguity in the choice of the solution ω which means $\Lambda_{X_M}\theta$ is well defined.

The results above and those in [2] provide the basic ingredients needed to extend by analogy the results in [7] and some of the results in [12] on the ring structure to the context of X_M-cohomology and the X_M-DN map. However, some results in Sections 4 and 6 are different and are specified here. We therefore omit the proof of the results below; full details are given in the first author's thesis [1].

Proposition 3.3.

(1) *$i^*\mathcal{H}^{\pm}_{X_M}(M) = \mathcal{E}^{\pm}_{X_M}(\partial M) + i^*\mathcal{H}^{\pm}_{X_M,N}(M)$, where $\mathcal{E}^{\pm}_{X_M}(\partial M) = \{\mathrm{d}_{X_M}\alpha \mid \alpha \in \Omega^{\mp}_G(\partial M)\}$.*
(2) *The operator Λ_{X_M} is nonnegative in the sense that the integral $\int_{\partial M}\theta \wedge \Lambda_{X_M}\theta$ is nonnegative for any $\theta \in \Omega^{\pm}_G(\partial M)$.*
(3) *Let $\omega \in \Omega^{\pm}_G(M)$ be a solution to the BVP (3.4) where $\theta \in \Omega^{\pm}_G(\partial M)$ is given. Then $\mathrm{d}_{X_M}\omega \in \mathcal{H}^{\mp}_{X_M}(M)$ and $\delta_{X_M}\omega = 0$.*
(4) *$\ker\Lambda_{X_M} = \operatorname{Ran}\Lambda_{X_M} = i^*\mathcal{H}_{X_M}(M)$, where $\mathcal{H}_{X_M} = \mathcal{H}^+_{X_M} \oplus \mathcal{H}^-_{X_M}$.*
(5) *The operator Λ_{X_M} satisfies the following relations*:

$$\Lambda_{X_M}\mathrm{d}_{X_M} = 0, \qquad \mathrm{d}_{X_M}\Lambda_{X_M} = 0, \qquad \Lambda^2_{X_M} = 0.$$

In this corollary, we introduce the X_M-Hilbert transform T_{X_M} which is of course the analogue of the usual Hilbert transform (see Section 5 in [7]) and it will be used in Section 6.

Corollary 3.4. *The operator $T_{X_M} := \mathrm{d}_{X_M}\Lambda^{-1}_{X_M} : i^*\mathcal{H}_{X_M}(M) \longrightarrow i^*\mathcal{H}_{X_M}(M)$ is well defined; i.e. the equation $\phi = \Lambda_{X_M}\theta$ has a solution θ for any $\phi \in i^*\mathcal{H}_{X_M}(M)$, and $\mathrm{d}_{X_M}\theta$ is uniquely determined by $\phi = \Lambda_{X_M}\theta$. In particular, $T_{X_M} : i^*\mathcal{H}^{\pm}_{X_M,N}(M) \longrightarrow i^*\mathcal{H}^{n-\pm}_{X_M,N}(M)$ and the operator $\mathrm{d}_{X_M}\Lambda^{-1}_{X_M}\mathrm{d}_{X_M} : \Omega_G(\partial M) \longrightarrow \Omega_G(\partial M)$ is well defined.*

The above construction and the results in [2] provide the essential ingredients needed to extend Theorem 4.2 of [7] (our Theorem 1.1) to the present context:

Theorem 3.5. *The Neumann X_M-trace spaces $i^*\mathcal{H}^{\pm}_{X_M,N}(M)$ can be completely determined from the boundary data $(\partial M, \Lambda_{X_M})$. In particular,*

$$\left(\Lambda_{X_M} - (\mp 1)^{n+1}\mathrm{d}_{X_M}\Lambda^{-1}_{X_M}\mathrm{d}_{X_M}\right)\Omega^{\pm}_G(\partial M) = i^*\mathcal{H}^{n-\mp}_{X_M,N}(M). \tag{3.5}$$

4. Λ_{X_M} operator, X_M-cohomology and equivariant cohomology

The following result is an extension of Theorem 1.2 to X_M-cohomology. We relate the dimension of $H^{\pm}_{X_M}(M)$ with the kernel of Λ_{X_M} as follows:

Theorem 4.1. *Let $\Lambda^{\pm}_{X_M}$ be the restriction of X_M-DN operator to $\Omega^{\pm}_G(\partial M)$. Then $\mathcal{E}^{\pm}_{X_M}(\partial M) \subseteq \ker\Lambda^{\pm}_{X_M}$ and*

$$\dim\left[\ker\Lambda^{\pm}_{X_M}/\mathcal{E}^{\pm}_{X_M}(\partial M)\right] \leqslant \min\left\{\dim\left(H^{\pm}_{X_M}(\partial M)\right), \dim\left(H^{\pm}_{X_M}(M)\right)\right\}. \tag{4.1}$$

Moreover, if every component of $F' = N(X_M)$ has a boundary then

$$\max\left\{\dim\left[\ker\Lambda^{\pm}_{X_M}/\mathcal{E}^{\pm}_{X_M}(\partial M)\right], \dim\left[\ker\Lambda^{\pm}/\mathcal{E}^{\pm}\left(\partial F'\right)\right]\right\} \leqslant \min\left\{\dim\left(H^{\pm}_{X_M}(\partial M)\right), \dim\left(H^{\pm}_{X_M}(M)\right)\right\}.$$

The proof of the first part follows the proof of Theorem 1.2 so we omit it (details are given in [1]). The second part follows by applying Theorem 1.2 to F'. It moreover refers implicitly to a possible relation between the dimensions of

$\ker \Lambda^{\pm}_{X_M}/\mathcal{E}^{\pm}_{X_M}(\partial M)$ and $\ker \Lambda^{\pm}/\mathcal{E}^{\pm}(\partial F')$ which needs to be discovered. This idea and others are under investigation in [1] which will help to extend many of the results of [12] to the style of X_M-cohomology.

To relate these inequalities to equivariant cohomology, one uses a result in [2] (essentially due to Atiyah and Bott), which asserts that if $F' = F := \mathrm{Fix}(G, M)$, then $\dim(H^{\pm}_{X_M}(M)) = \operatorname{rank} H^{\pm}_G(M)$—see, Theorem 6.1 below. Hence we conclude that under this assumption, the right-hand side of the inequalities above can be replaced by $\min\{\operatorname{rank} H^{\pm}_G(\partial M), \operatorname{rank} H^{\pm}_G(M)\}$.

5. Recovering X_M-cohomology from the boundary data $(\partial M, \Lambda_{X_M})$

In this section, we continue extending the results of Belishev–Sharafutdinov and Shonkwiler's Theorem 1.3 on recovering the de Rham cohomology groups and ring structure from the boundary data $(\partial M, \Lambda)$, to the context of absolute and relative X_M-cohomology and their concrete realizations $\mathcal{H}^{\pm}_{X_M,N}(M)$ and $\mathcal{H}^{\pm}_{X_M,D}(M)$.

5.1. Recovering the long exact X_M-cohomology sequence of $(M, \partial M)$

We show that the data $(\partial M, \Lambda_{X_M})$ determines the long exact X_M-cohomology sequence of the pair $(M, \partial M)$.

Since the vector field X_M which we are considering is always tangent to the boundary ∂M, we can also define X_M-cohomology on ∂M, that is $H^{\pm}_{X_M}(\partial M)$. Hence, from the definitions of the absolute and relative X_M-cohomology, we have the following exact X_M-cohomology sequence of the pair $(M, \partial M)$ as follows:

$$\cdots \xrightarrow{\partial^*} H^{\pm}_{X_M}(M, \partial M) \xrightarrow{\rho^*} H^{\pm}_{X_M}(M) \xrightarrow{i^*} H^{\pm}_{X_M}(\partial M) \xrightarrow{\partial^*} H^{\mp}_{X_M}(M, \partial M) \xrightarrow{\rho^*} \cdots. \tag{5.1}$$

However, Theorem 3.5 proves that we can determine the space $i^*\mathcal{H}^{\pm}_{X_M,N}(M)$ from the boundary data and Corollary 2.3 gives $i^*\mathcal{H}^{\pm}_{X_M,N}(M) \cong H^{\pm}_{X_M}(M)$ and $i^*\mathcal{H}^{n-\pm}_{X_M,N}(M) \cong H^{\pm}_{X_M}(M, \partial M)$. This says that the additive absolute and relative X_M-cohomology are completely determined by $(\partial M, \Lambda_{X_M})$.

So, if the boundary data $(\partial M, \Lambda_{X_M})$ is given, we can construct the sequence

$$\cdots \xrightarrow{\bar{\partial}^*} i^*\mathcal{H}^{n-\pm}_{X_M,N}(M) \xrightarrow{\bar{\rho}^*} i^*\mathcal{H}^{\pm}_{X_M,N}(M) \xrightarrow{\bar{i}^*} H^{\pm}_{X_M}(\partial M) \xrightarrow{\bar{\partial}^*} i^*\mathcal{H}^{n-\mp}_{X_M,N}(M) \xrightarrow{\bar{\rho}^*} \cdots \tag{5.2}$$

where we define the operators of sequence (5.2) by the following formulas:

(1) $\bar{i}^*\theta = [\theta]_{(X_M,\partial M)}$; if $\theta \in i^*\mathcal{H}^{\pm}_{X_M,N}$ then θ is X_M-closed because i^* and d_{X_M} commute.
(2) Using Corollary 3.4 we set, $\bar{\rho}^*\theta = -(\pm 1)^{n+1} T_{X_M}\theta$, $\forall \theta \in i^*\mathcal{H}^{n-\pm}_{X_M,N}$.
(3) Let $\theta \in \Omega_G(\partial M)$ be X_M-closed. Based on Theorem 3.5, $\Lambda_{X_M}\theta = (\Lambda_{X_M} - (\mp 1)^{n+1}\mathrm{d}_{X_M}\Lambda^{-1}_{X_M}\mathrm{d}_{X_M})\theta$. Hence, we set

$$\bar{\partial}^*[\theta]_{(X_M,\partial M)} = (\mp 1)^{n+1}\Lambda_{X_M}\theta, \quad \forall [\theta]_{(X_M,\partial M)} \in H^{\pm}_{X_M}(\partial M).$$

More concretely, our goal is then to recover sequence (5.1) from sequence (5.2). It means that we should prove that the following diagram (5.3) is commutative:

$$\begin{tikzcd}
\cdots \arrow[r, "\bar{\partial}^*"] & i^*\mathcal{H}^{n-\pm}_{X_M,N}(M) \arrow[r, "\bar{\rho}^*"] \arrow[d, "h"] & i^*\mathcal{H}^{\pm}_{X_M,N}(M) \arrow[r, "\bar{i}^*"] \arrow[d, "f"] & H^{\pm}_{X_M}(\partial M) \arrow[r, "\bar{\partial}^*"] \arrow[d, "\iota"] & i^*\mathcal{H}^{n-\mp}_{X_M,N}(M) \arrow[r, "\bar{\rho}^*"] \arrow[d, "h"] & \cdots \\
\cdots \arrow[r, "\partial^*"] & H^{\pm}_{X_M}(M, \partial M) \arrow[r, "\rho^*"] & H^{\pm}_{X_M}(M) \arrow[r, "i^*"] & H^{\pm}_{X_M}(\partial M) \arrow[r, "\partial^*"] & H^{\mp}_{X_M}(M, \partial M) \arrow[r, "\rho^*"] & \cdots
\end{tikzcd} \tag{5.3}$$

where ι is the identity operator while f and h are given in Corollary 2.3. Indeed, one can prove the commutativity of the diagram by a method similar to that given in [7] but in terms of the operators d_{X_M} and δ_{X_M}, see [1] for details.

Actually, the above construction proves that the data $(\partial M, \Lambda_{X_M})$ recovers sequence (5.1) of the pair $(M, \partial M)$ up to an isomorphism (i.e. up to the maps f and h) from the sequence (5.2).

5.2. Recovering the ring structure of the real X_M-cohomology

We consider the following question: *can the multiplicative ring structure of the real absolute and relative X_M-cohomology be recovered from the boundary data $(\partial M, \Lambda_{X_M})$?*

First of all, we consider the mixed cup product $\bar{\cup}$ between the absolute and relative X_M-cohomology as follows:

$$\bar{\cup} : H^{\pm}_{X_M}(M) \times H^{\pm}_{X_M}(M, \partial M) \longrightarrow H^{\pm}_{X_M}(M, \partial M),$$
$$[\alpha]_{(X_M,M)} \,\bar{\cup}\, [\beta]_{(X_M,M,\partial M)} = [\alpha \wedge \beta]_{(X_M,M,\partial M)}.$$

It is easy to check that $\bar{\cup}$ is a well-defined map. In addition, in [2] we prove that any absolute or relative X_M-cohomology classes contain a unique Neumann or Dirichlet X_M-harmonic field respectively. Hence, we can regard any absolute (relative)

X_M-cohomology class as a Neumann (Dirichlet) X_M-harmonic field. But $[\alpha]_{(X_M,M)} \bar{\cup} [\beta]_{(X_M,M,\partial M)} = [\alpha\wedge\beta]_{(X_M,M,\partial M)}$ is a relative X_M-cohomology class, so there exists a unique Dirichlet X_M-harmonic field $\eta \in \mathcal{H}^{\pm}_{X_M,D}(M)$ such that $[\alpha\wedge\beta]_{(X_M,M,\partial M)} = [\eta]_{(X_M,M,\partial M)}$, i.e.

$$\alpha\wedge\beta = \eta + \mathrm{d}_{X_M}\xi \in \mathcal{H}^{\pm}_{X_M,D}(M) \oplus \mathcal{E}^{\pm}_{X_M}(M) \tag{5.4}$$

for some $\xi \in \Omega^{\mp}_{G,D}(M)$. However, it follows from Corollary 2.3 that

$$H^{\pm}_{X_M}(M,\partial M) \cong H^{n-\pm}_{X_M}(M) \cong i^*\mathcal{H}^{n-\pm}_{X_M,N}(M).$$

According to our illustrations above we know that an absolute X_M-cohomology class $[\alpha]_{(X_M,M)} \in H^{\pm}_{X_M}(M)$ and relative X_M-cohomology classes $[\beta]_{(X_M,M,\partial M)}, [\alpha\wedge\beta]_{(X_M,M,\partial M)} \in H^{\pm}_{X_M}(M,\partial M)$ are represented by the Neumann X_M-harmonic field $\alpha \in \mathcal{H}^{\pm}_{X_M,N}(M)$ and the Dirichlet X_M-harmonic fields $\beta,\eta \in \mathcal{H}^{\pm}_{X_M,D}(M)$ respectively, such that they correspond, respectively, to forms on the boundary by setting

$$\phi = i^*\alpha \in i^*\mathcal{H}^{\pm}_{X_M,N}(M), \qquad \psi = i^*\star\beta \in i^*\mathcal{H}^{n-\pm}_{X_M,N}(M), \qquad \vartheta = i^*\star\eta \in i^*\mathcal{H}^{n-\pm}_{X_M,N}(M).$$

Following [12], our answer to the above question will only be partial, in the sense that we will not consider all the classes of the relative X_M-cohomology, but will just consider the *boundary portion*, denoted $BH^{\pm}_{X_M}(M,\partial M)$, of $H^{\pm}_{X_M}(M,\partial M)$. This boundary subspace is defined to be [2],

$$BH^{\pm}_{X_M}(M,\partial M) = \mathrm{im}\big[\partial^* : H^{\mp}_{X_M}(\partial M) \longrightarrow H^{\pm}_{X_M}(M,\partial M)\big].$$

Here ∂^* is the standard construction in the long exact sequence (5.1): given an X_M-closed form λ on ∂M, let $\tilde{\lambda}$ be an extension form on M. Then $\mathrm{d}_{X_M}\tilde{\lambda}$ defines a well-defined element of $H^{\pm}_{X_M}(M,\partial M)$ denoted $\partial^*\lambda$. This boundary portion is therefore the image of $H^{\pm}_{X_M}(\partial M)$ inside $H^{\mp}_{X_M}(M,\partial M)$ in this long exact sequence.

In [2], we prove that $H^{\pm}_{X_M}(M,\partial M) \cong \mathcal{H}^{\pm}_{X_M,D}(M)$. Hence, on translation into the language of X_M-harmonic fields, we can identify

$$BH^{\pm}_{X_M}(M,\partial M) \cong \mathcal{BH}^{\pm}_{X_M,D}$$

where $\mathcal{BH}^{\pm}_{X_M,D} = \mathcal{H}^{\pm}_{X_M,D}(M) \cap \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}$ is called the boundary subspace of $\mathcal{H}^{\pm}_{X_M,D}(M)$. Clearly, Hodge star $\star$ gives

$$\mathcal{BH}^{n-\pm}_{X_M,N}(M) := \star\mathcal{BH}^{\pm}_{X_M,D}$$

where $\mathcal{BH}^{n-\pm}_{X_M,N}(M) = \mathcal{H}^{n-\pm}_{X_M,N}(M) \cap \mathcal{H}^{n-\pm}_{X_M,\mathrm{co}}$ is the boundary subspace of $\mathcal{H}^{n-\pm}_{X_M,N}(M)$. Using this fact together with Corollary 2.3 we conclude that $BH^{\pm}_{X_M}(M,\partial M) \cong i^*\star\mathcal{BH}^{\pm}_{X_M,D}$.

The above constructions allow us to extend Shonkwiler's map [12] to the context of Λ_{X_M} in order to define the following map with notation as above:

$$\begin{aligned} &\bar{\cup} : i^*\mathcal{H}^{\pm}_{X_M,N}(M) \times i^*\mathcal{H}^{n-\pm}_{X_M,N}(M) \longrightarrow H^{\pm}_{X_M}(M,\partial M), \\ &\phi \,\bar{\cup}_{X_M}\, \psi = \Lambda_{X_M}\big(\pm\phi\wedge\Lambda^{-1}_{X_M}\psi\big). \end{aligned} \tag{5.5}$$

By using the same method as [12] together with Definition 3.2 we deduce that $\bar{\cup}_{X_M}$ is well defined. Now, we can extend Shonkwiler's Theorem 1.3 to the style above.

Theorem 5.1. *The boundary data $(\partial M, \Lambda_{X_M})$ completely determines the mixed cup product structure of the X_M-cohomology when the relative X_M-cohomology classes come from the boundary subspace, i.e. if $(\alpha,\beta) \in \mathcal{H}^{\pm}_{X_M,N}(M) \times \mathcal{BH}^{\pm}_{X_M,D}(M)$ such that $\alpha\wedge\beta = \eta + d_{X_M}\xi \in \mathcal{H}^{\pm}_{X_M,D}(M) \oplus \mathcal{E}^{\pm}_{X_M}(M)$ then*

$$i^*\star\eta = \Lambda_{X_M}\big(\pm\phi\wedge\Lambda^{-1}_{X_M}\psi\big)$$

where $\phi = i^*\alpha$ and $\psi = i^*\star\beta$. In fact one shows the commutativity of the following diagram:

$$\begin{array}{ccc} i^*\mathcal{H}^{\pm}_{X_M,N}(M) \times i^*\star\mathcal{BH}^{\pm}_{X_M,D}(M) & \xrightarrow{\bar{\cup}_{X_M}} & i^*\star\mathcal{BH}^{\pm}_{X_M,D}(M) \\ \big\downarrow{\scriptstyle (f,h)} & & \big\downarrow{\scriptstyle h} \\ H^{\pm}_{X_M}(M) \times BH^{\pm}_{X_M}(M,\partial M) & \xrightarrow{\bar{\cup}} & BH^{\pm}_{X_M}(M,\partial M), \end{array} \tag{5.6}$$

where f and h are given in Corollary 2.3.

Proof. Our goal is to show that $\forall(\phi,\psi)=(i^*\alpha, i^*\star \mathrm{d}_{X_M}\beta_1)\in i^*\mathcal{H}^{\pm}_{X_M,N}(M)\times i^*\star\mathcal{BH}^{\pm}_{X_M,D}(M)$ one has

$$(h\circ \bar{\cup}_{X_M})\big(i^*\alpha, i^*\star \mathrm{d}_{X_M}\beta_1\big)=\big(\bar{\cup}\circ(f,h)\big)\big(i^*\alpha, i^*\star \mathrm{d}_{X_M}\beta_1\big). \tag{5.7}$$

The left-hand side gives

$$h\big(\bar{\cup}_{X_M}\big(i^*\alpha, i^*\star \mathrm{d}_{X_M}\beta_1\big)\big)=h\big(\Lambda_{X_M}\big(\pm\phi\wedge\Lambda^{-1}_{X_M}\psi\big)\big) \tag{5.8}$$

while the right-hand side together with Eq. (5.4) and Corollary 2.3 give

$$\begin{aligned}\bar{\cup}\big(\big(f\big(i^*\alpha\big),h\big(i^*\star \mathrm{d}_{X_M}\beta_1\big)\big)\big)&=\bar{\cup}\big([\alpha]_{(X_M,M)},[\star\star \mathrm{d}_{X_M}\beta_1]_{(X_M,M,\partial M)}\big)\\&=\big[\star\star(\alpha\wedge \mathrm{d}_{X_M}\beta_1)\big]_{(X_M,M,\partial M)}\\&=[\star\star\eta]_{(X_M,M,\partial M)}\\&=h\big(i^*\star\eta\big).\end{aligned} \tag{5.9}$$

We now need to prove that the right-hand sides of Eqs. (5.8) and (5.9) are equal. This will be the case if

$$i^*\star\eta=\Lambda_{X_M}\big(\pm\phi\wedge\Lambda^{-1}_{X_M}\psi\big). \tag{5.10}$$

The method of Shonkwiler [12] used to prove Theorem 1.3 extends to our setting by combining with results in [2], such as the X_M-Hodge–Morrey decomposition theorem (full details are given in [1]). □

6. Conclusions

(1) The key point used to recover the free part of the relative and absolute equivariant cohomology groups from the boundary data $(\partial M,\Lambda_{X_M})$ is the following theorem which is essentially Atiyah and Bott's localization theorem.

Theorem 6.1. *([2]) Let $X\in\mathfrak{g}$ (the Lie algebra of G) and let $F'=N(X_M)$. The inclusion $j_X: F'\hookrightarrow M$ induces the following isomorphisms*:

(1) $H^{\pm}_{X_M}(M)\cong H^{\pm}(F')$,
(2) $H^{\pm}_{X_M}(M,\partial M)\cong H^{\pm}(F',\partial F')$.

Moreover, if $N(X_M)=F:=\mathrm{Fix}(G,M)$ then $\dim H^{\pm}(F,\partial F)=\operatorname{rank} H^{\pm}_G(M,\partial M)$ *and* $\dim H^{\pm}(F)=\operatorname{rank} H^{\pm}_G(M)$.

Now, combining the above theorem with Theorem 3.5 and Corollary 2.3, we deduce

Theorem 6.2.

$$H^{\pm}_{X_M}(M,\partial M)\cong\big(\Lambda_{X_M}-(\pm1)^{n+1}\mathrm{d}_{X_M}\Lambda^{-1}_{X_M}\mathrm{d}_{X_M}\big)\Omega^{\mp}_G(\partial M)\cong H^{\pm}\big(F',\partial F'\big)$$

and

$$H^{\pm}_{X_M}(M)\cong\big(\Lambda_{X_M}-(\pm1)^{n+1}\mathrm{d}_{X_M}\Lambda^{-1}_{X_M}\mathrm{d}_{X_M}\big)\Omega^{n-\mp}_G(\partial M)\cong H^{\pm}\big(F'\big).$$

Since the Neumann X_M-harmonic fields are uniquely determined by their Neumann X_M-trace (Corollary 2.3) which is in turn determined by the boundary data $(\partial M,\Lambda_{X_M})$ (Theorem 3.5), this means we can conclude, by using X_M-Poincaré–Lefschetz duality of [2], that we can realize the relative and absolute X_M-cohomology groups (and hence in some sense the free part of the relative and absolute equivariant cohomology groups) as particular subspaces of invariant differential forms on ∂M and they are not just determined abstractly from the generalized boundary data.

(2) We can apply Theorem 1.1 to the manifolds $F'=N(X_M)$ with boundary $\partial F'$. Since G acts on F' the induced action on each $H^{\pm}(F')$ is trivial. Now, we can use Theorem 6.2 to exploit the connection between Belishev and Sharafutdinov's boundary data on $\partial F'$ (i.e. $(\partial F',\Lambda)$) and ours on ∂M (i.e. $(\partial M,\Lambda_{X_M})$). More concretely, we have the following.

Theorem 6.3. *If every component of F' has a boundary, then*

$$\big(\Lambda_{X_M}-(\mp1)^{n+1}\mathrm{d}_{X_M}\Lambda^{-1}_{X_M}\mathrm{d}_{X_M}\big)\Omega^{\pm}_G(\partial M)\cong\big(\Lambda-(\mp1)^{n+1}\mathrm{d}\Lambda^{-1}\mathrm{d}\big)\Omega^{\pm}\big(\partial F'\big).$$

This means that the boundary data $(\partial F',\Lambda)$ can be determined from the boundary data $(\partial M,\Lambda_{X_M})$ and vice versa. In this setting, it follows that since the de Rham cohomology groups of $(F',\partial F')$ are determined by $(\partial F',\Lambda)$ (Theorem 1.1), then the $\pm$ de Rham cohomology groups of $(F',\partial F')$ are also determined by $(\partial M,\Lambda_{X_M})$.

(3) When M has no boundary, Witten proves in [13] that $H^{\pm}_{K}(M) \cong H^{\pm}(F')$ where K is a Killing vector field (our X_M) on M and he shows how the K-cohomology and the isomorphism above are useful in quantum field theory and other mathematical and physical applications. However, when $\partial M \neq \emptyset$, the extended isomorphism is provided by Theorem 6.1 above which gives insight that the extension for other results of Witten [13] are possible. In this light, Theorem 6.2 suggests that Λ_{X_M} may also be relevant to quantum field theory and following Witten, possibly to other mathematical and physical interpretations. This shows that Λ_{X_M} may be interesting in its own right.

Finally, it is worth considering the following topological problem: *Can the torsion part of the absolute and relative equivariant cohomology groups be completely recovered from the boundary data* $(\partial M, \Lambda_{X_M})$? (Here torsion is meant as a module over the ring of polynomials on $\mathfrak{g}$—the standard Cartan model: some torsion information is available from Theorems 6.1 and 6.2 when X is in an isotropy subalgebra, but not all.) Answering this question will indeed complete the picture of the role the boundary data $(\partial M, \Lambda_{X_M})$ plays in the story of the equivariant cohomology of manifolds.

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